Holographic correlators and emergent Parisi-Sourlas supersymmetry

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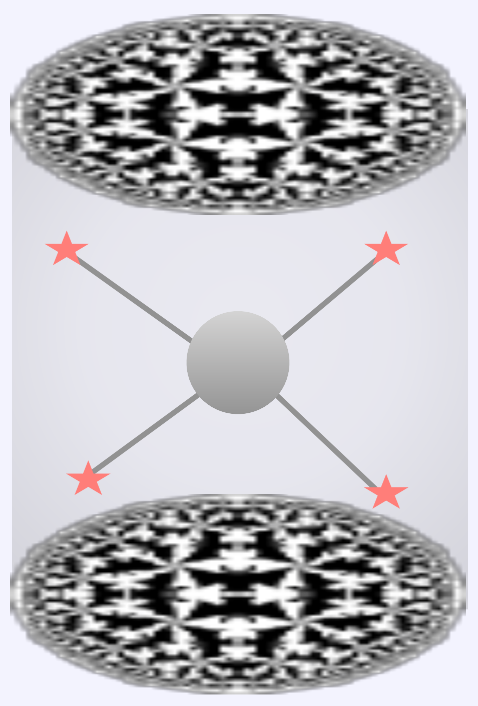
Holographic correlators — why study them?

AdS/CFT duality:



$AdS_{d+1} \times S^q$

$\Leftrightarrow$

$\mathbb{R}^{d-1,1} = \partial AdS_{d+1}$

- String/M-theory on $AdS_{d+1} \times S^q$ is dual to superconformal field theory on $\mathbb{R}^{d-1,1}$;
- Most basic observables of AdS/CFT correspondence: CFT correlators are scattering amplitudes in the bulk AdS space;
- The most tractable limit in the bulk is when the bulk theory becomes a weakly coupled supergravity theory.

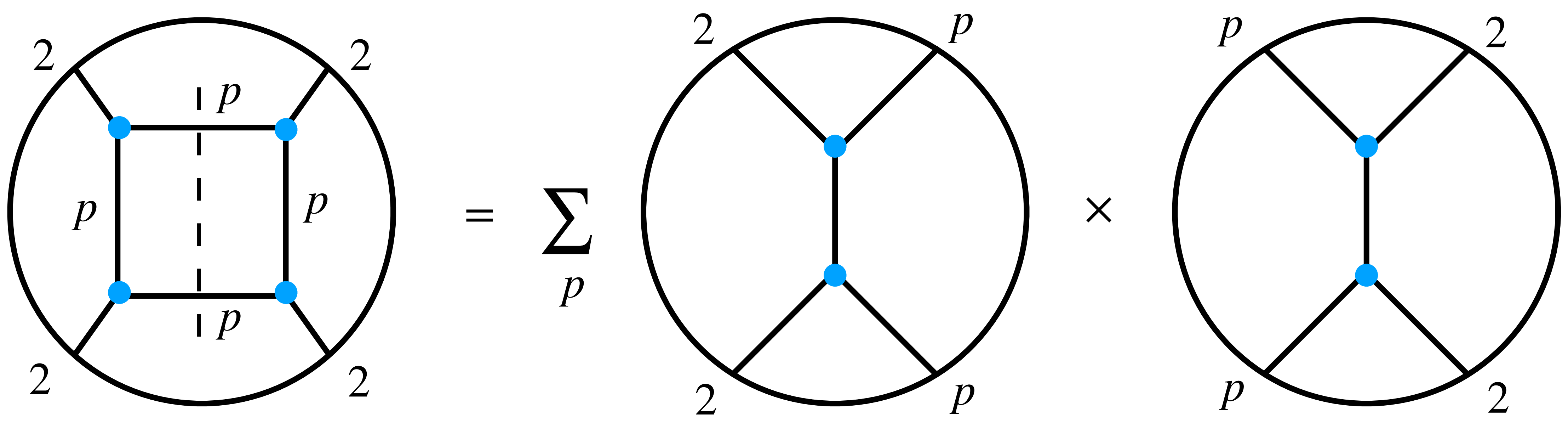
Motivations:

1. Use weakly coupled bulk theory to learn about strongly coupled boundary theory.
   - Unprotected quantities (e.g. OPE coefficients and anomalous dimensions of double-trace operators) are difficult to compute at strong coupling using field theory methods.
   - Non-Lagrangian theories (e.g., the 6d (2,0) theory): using AdS/CFT is the only way.

Holographic correlators — why study them?

2. Use conformal bootstrap techniques to learn about quantum gravity (trees → loops).

A direct bulk computation of correlators at one-loop level is technically too difficult, but also unnecessary. The fact that the dual boundary theory is a CFT allows us to use bootstrap techniques to compute loop correlators, after inputting the tree-level data. This can be viewed as an AdS version of the unitarity method in flat space [Aharony, Alday, Bissi, Perlmutter]. Holographic correlators provide a concrete setup to study AdS gravity at quantum level [Alday, Bissi; Aprile, Drummond, Heslop, Paul; Alday, Caron-Huot; Caron-Huot, Trinh; Alday, Bissi, Perlmutter; Alday, XZ; Bissi, Fardelli, Georgoudis; Alday, Chester, Raj ...]



This can also be extended beyond two-derivative supergravity, and incorporate higher-derivative corrections from string/M-theory by combining with techniques such as supersymmetric localization [Binder, Chester, Green, Pufu, Wang, Wen; Alday, Bissi, Perlmutter; Drummond, Nandan, Paul, Rigatos, Santagata; Abl, Heslop, Lipstein....]. Studying these correlators provides precision tests of the AdS/CFT correspondence.

Holographic correlators — why study them?

3. Connection with flat space scattering amplitudes.

- The flat space scattering amplitude program is highly successful: efficient methods to compute amplitudes without using Feynman diagrams; rich mathematical structures (color/kinematic duality, the positive grassmanian, CHY construction, etc).
- If these progress are teaching us new ways to think about QFTs, then the lessons should also extend to curved spacetime. AdS space is the simplest curved spacetime, and the perfect playground for exploring such extensions.
- Flat space amplitudes teach us a lot about the symmetries of the theory: dual conformal symmetry, duality with Wilson loops, etc. What symmetry properties can we learn about theories in AdS?

What are we computing?

What are the theories? We will focus on theories with the maximal amount of supersymmetry (i.e., sixteen Poincaré supercharges) in $d > 2$ to have most analytic power. This means have:

IIB sugra on AdS5xS5 = 4d N=4 SYM, 11D sugra on AdS4xS7 = 3d N=8 ABJM, 11D sugra on AdS7xS4=6d N=(2,0)

What are the operators/particles? Reduction on the sphere gives rise to infinite Kaluza-Klein towers of component fields, organized into superconformal multiplets. We will scatter the super primaries, which in the bulk are scalar fields.

CFT side	SUGRA side
one-half BPS operator : $\mathcal{O}_k^{I_1\ldots I_k}$, $I = 1,2,\ldots,q+1$, $k = 2,3,\ldots$,	scalar field : $s_k^{I_1\ldots I_k}$
R-symmetry $SO(q+1)$ irrep: rank-k symmetric traceless	S^q angular momentum: k
conformal dimension: $\Delta = \epsilon k$, $\epsilon = \frac{d-2}{2}$	squared mass: $m^2 = \Delta(\Delta - d)$

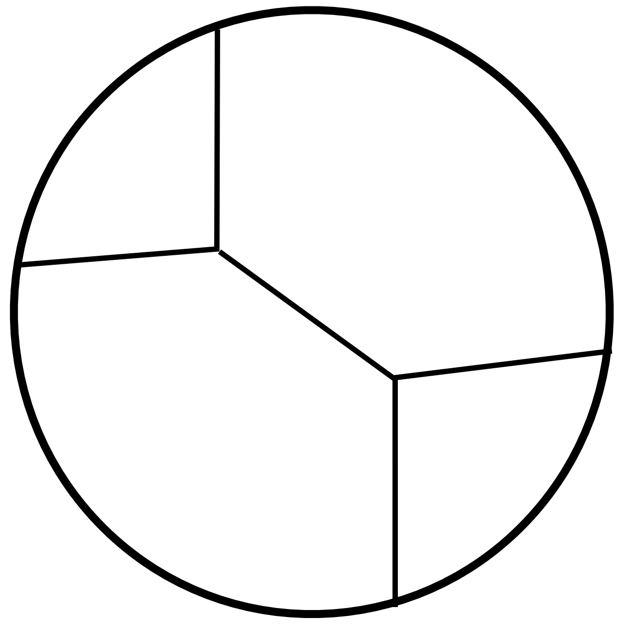
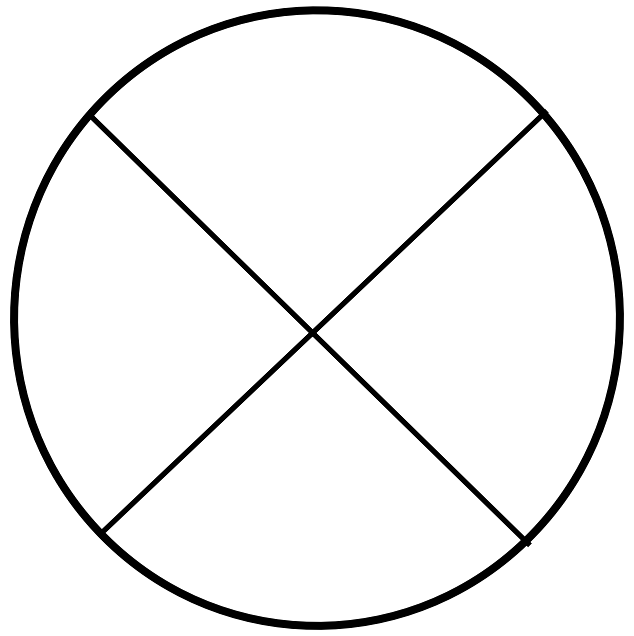
Example: $k = 2$ is the stress tensor multiplet.

Two- and three-point functions are trivial. We compute the four-point functions $\langle \mathcal{O}_{k_1}\mathcal{O}_{k_2}\mathcal{O}_{k_3}\mathcal{O}_{k_4}\rangle$. Infinitely many of them!!

How were they computed?

The traditional method is a brute force diagrammatic expansion inside AdS space.

1. Feynman rules in AdS
   (a) propagators;
   (b) vertices: cubic and quartic;
   (c) integrate over AdS.
2. Enumerate all diagrams and evaluate them.
3. Add them up.

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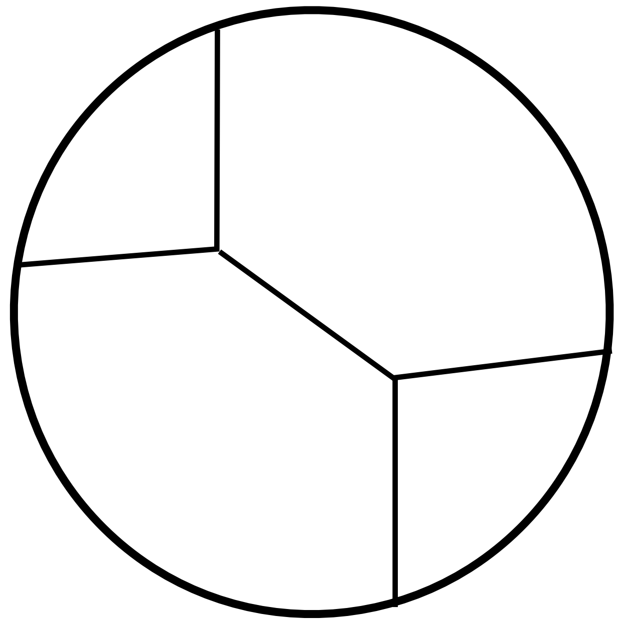
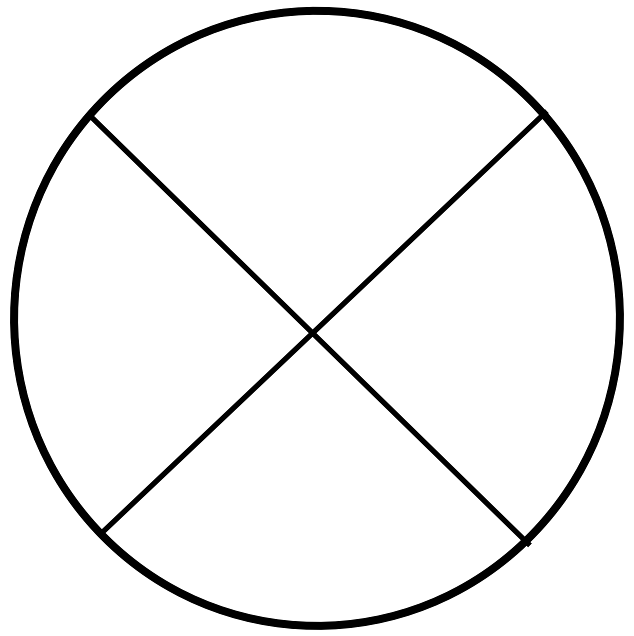
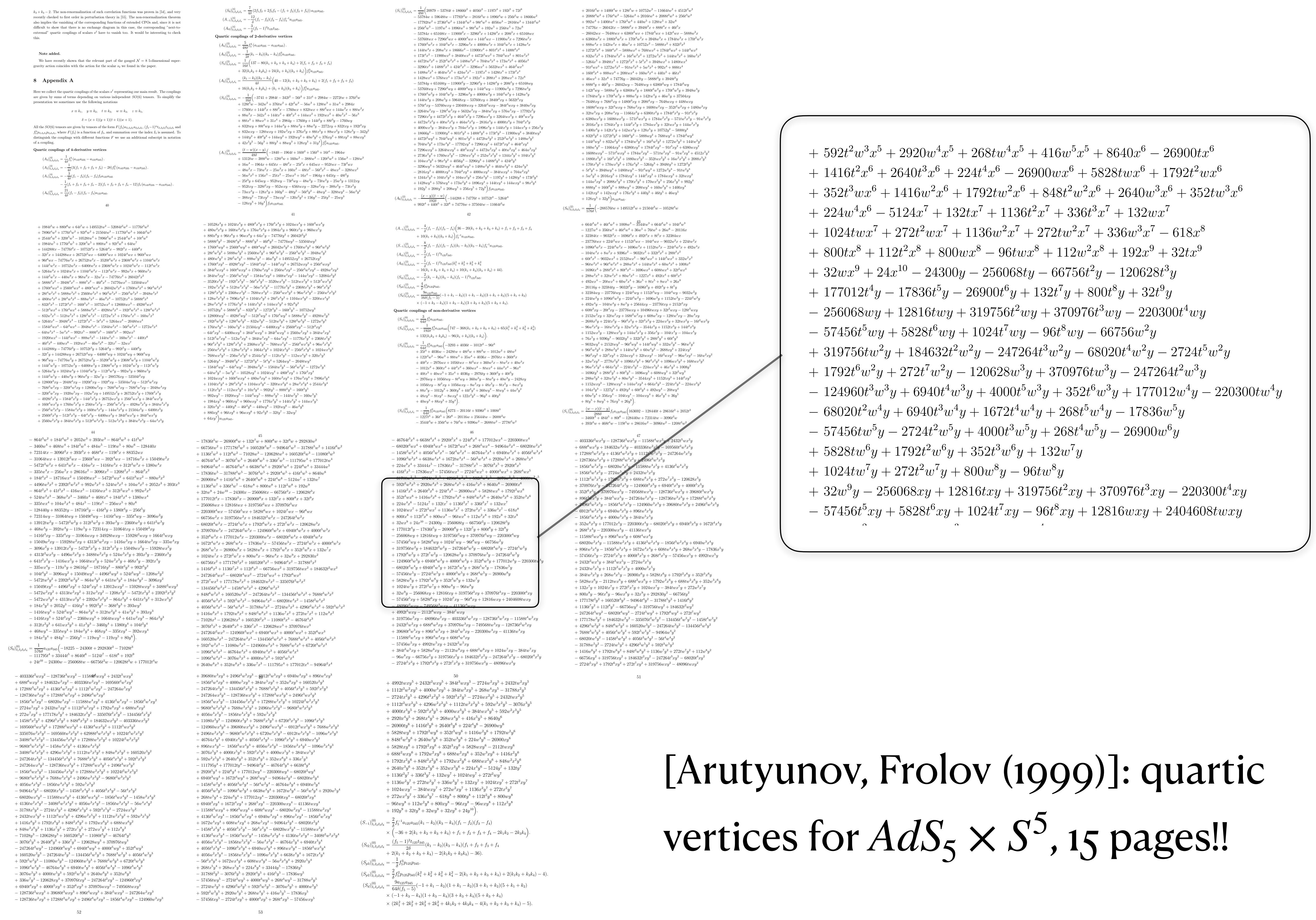






[Arutyunov, Frolov (1999)]: quartic vertices for $AdS_5 \times S^5$, 15 pages!!

Traditional method: seems like a recipe for disasters....

How were they computed?

- The traditional method is extremely cumbersome to use for these reasons, and only a handful results were computed in the old literature*. Most of them are in $AdS_5 \times S^5$:
  - axion-dilation four-point function [D'Hoker, Freedman, Mathur, Matusis, Rastelli],
  - $\langle 2222 \rangle$, $\langle 3333 \rangle$, $\langle 4444 \rangle$ [Arutyunov, Dolan, Frolov, Osborn, Sokatchev],
  - next-to-next-to-extremal $\langle 2+k, 2+k, k-n, k+n \rangle$ [Berdichevsky, Naaijkens, Uruchurtu],
  - a conjecture for $\langle pppp \rangle$ [Dolan, Nirschl, Osborn].
- One example in $AdS_7 \times S^4$: $\langle 2222 \rangle$ [Arutyunov, Sokatchev]. Nothing else for other backgrounds.
- The case for $AdS_4 \times S^7$ is especially hard: the exchange diagrams cannot be evaluated in a closed form in position space!

- Huge answers with growing complexity: the general structure is very unclear. What are the organizing principles for these holographic objects?

* The traditional method was further streamlined for the $AdS_5 \times S^5$ case in 2018 by Arutyunov, Frolov, Klabbers, Savin, and more examples were computed.

Bootstrap it!

There is a lot of recent progress using bootstrap ideas [Rastelli, XZ '16]. One lesson is that holographic correlators have very rigid structures: they are fixed by superconformal symmetry and consistency conditions! I will review the bootstrap approaches and results. We will also see intrinsic limitations, which prompt the new developments I will describe later.

Let us first discuss superconformal symmetry.

$$\mathcal{O}_k(x,t) = \mathcal{O}_k^{I_1,\ldots,I_k}(x)t_{I_1}\ldots t_{I_k}\,, \quad t\cdot t=0 \qquad G_{k_1k_2k_3k_4}(x_i,t_i) = \langle \mathcal{O}_{k_1}\mathcal{O}_{k_2}\mathcal{O}_{k_3}\mathcal{O}_{k_4}\rangle$$

$$G(x_i,t_i) = \prod_{i<j}\left(\frac{t_{ij}}{x_{ij}^{2\epsilon}}\right)^{\gamma_{ij}^0}\left(\frac{t_{12}t_{34}}{x_{12}^{2\epsilon}x_{34}^{2\epsilon}}\right)^{\mathcal{E}}\mathcal{G}(U,V;\sigma,\tau) \qquad U=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}\,, \quad V=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}\,, \quad \sigma=\frac{t_{13}t_{24}}{t_{12}t_{34}}\,, \quad \tau=\frac{t_{14}t_{23}}{t_{12}t_{34}}$$

$\mathcal{E}$ is the extremality, defined as $\mathcal{E}=\frac{1}{2}(k_1+k_2+k_3-k_4)$ if $k_1+k_4\geq k_2+k_3$ and $\mathcal{E}=k_1$ otherwise, where we have assumed that $k_1\leq k_2\leq k_3\leq k_4$. $\mathcal{G}$ is a degree-$\mathcal{E}$ polynomial in σ and τ, and $\mathcal{E}$ determines the complexity of the correlator.

Superconformal symmetry implies the following superconformal Ward identity

$$(z\partial_z-\epsilon\alpha\partial_\alpha)\mathcal{G}(z,\bar{z};\alpha,\bar{\alpha})\Big|_{\alpha=1/z}=0 \qquad \epsilon=\frac{d-2}{2}$$

$$U=z\bar{z}\,, \qquad V=(1-z)(1-\bar{z})\,, \qquad \sigma=\alpha\bar{\alpha}\,, \qquad \tau=(1-\alpha)(1-\bar{\alpha})$$

Bootstrap it!

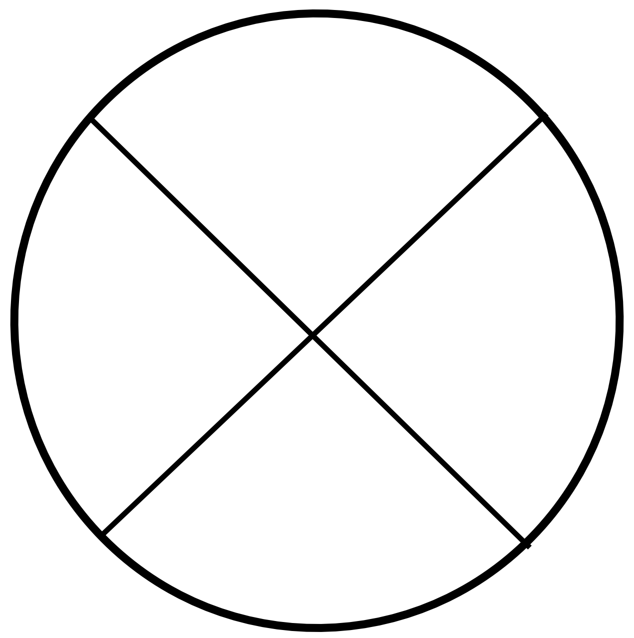
To have better control of the analytic structure, it's convenient to use the Mellin representation [Mack, Penedones].

$$\mathcal{G}_{\rm tree}(U,V;\sigma,\tau)=\int_{-i\infty}^{i\infty}\frac{dsdt}{(4\pi i)^2}U^{\frac{s}{2}-a_s}V^{\frac{t}{2}-a_t}\mathcal{M}(s,t;\sigma,\tau)\,\Gamma_{\{k_i\}}$$

$$\Gamma_{\{k_i\}}=\Gamma[\frac{\epsilon(k_1+k_2)-s}{2}]\Gamma[\frac{\epsilon(k_3+k_4)-s}{2}]\Gamma[\frac{\epsilon(k_1+k_4)-t}{2}]\Gamma[\frac{\epsilon(k_2+k_3)-t}{2}]\Gamma[\frac{\epsilon(k_1+k_3)-u}{2}]\Gamma[\frac{\epsilon(k_2+k_4)-u}{2}]$$

$$s+t+u=\epsilon(k_1+k_2+k_3+k_4)$$

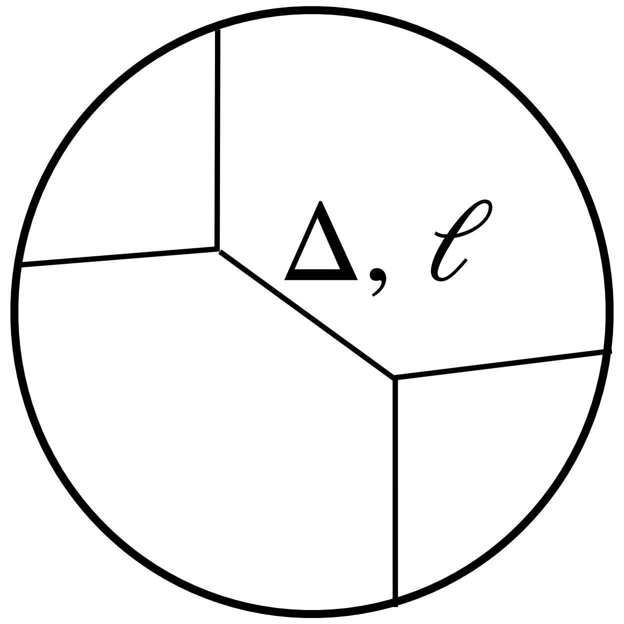
Individual Witten diagrams have very simple analytic structure in Mellin space



$\Rightarrow$ degree-L polynomial

$2L$-derivative

Complicated Appell F4



$$\Rightarrow \sum_{m=0}^{\infty}\frac{\mathcal{Q}_{m,\ell}(t,u)}{s-\underbrace{(\Delta-\ell)}_{\text{conformal twist}}-2m}+P_{\ell-1}(s,t)$$

- $P_{\ell-1}$ can be set to any value by tuning the cubic couplings.
- Poles truncate for $AdS_5\times S^5$ and $AdS_7\times S^4$, but not for $AdS_4\times S^7$. Truncation is a consistency with $1/N$ expansion.

Bootstrap it!

Three complementary bootstrap methods [Rastelli, XZ]:

1. **Position space method** [1608.06624, 1710.05923]:

   Mimics the traditional approach, but replaces vertices with unfixed coefficients in the sum over diagrams. The coefficients are then fixed by the WIs.

2. **Algebraic bootstrap method** [1608.06624, 1710.05923]:

   We first solve the WIs in position space. The correlator takes the form

   $$\mathcal{G} = \mathcal{G}_0 + \mathcal{D} \circ \mathcal{H}$$

   $\mathcal{G}_0$: protected; $\mathcal{H}$: reduced correlator (dynamical)

   We can then define a reduced Mellin amplitude $\widetilde{\mathcal{M}}$ from $\mathcal{H}$ to have $\mathcal{M} = \widehat{\mathcal{D}} \circ \widetilde{\mathcal{M}}$, and then impose analytic structures, Bose symmetry and asymptotic behavior on $\mathcal{M}$ to formulate an algebraic bootstrap problem. Solved all $AdS_5 \times S^5$ four-point functions in this way!

3. **Mellin superconformal Ward identities** [1712.02800]:

   Basically method 1, but in Mellin space thanks to a nice trick to translate the WI into Mellin space.

	Pros	Cons
1.	Proof of principle No vertices needed!	Need diagrams Cumbersome for higher weight correlators
2.	No diagrams needed! Solve all correlators all in one go! Can be formulated for $AdS_7 \times S^4$ as well [1712.02788]	$\widetilde{\mathcal{M}}$ have more obscure analytic structures. The $AdS_7 \times S^4$ bootstrap problem is hard to solve in general. Doesn't apply to $AdS_4 \times S^7$ because $\mathcal{D}$ is non local.
3.	Works for any spacetime dimension First correlator in $AdS_4 \times S^7$	Also case by case, difficult to apply for general correlators

Bootstrap it!

The bootstrap methods are successful only in limited domains, and do not provide us with a general picture.

supergravity theory	background	boundary theory	
IIB	$AdS_5 \times S^5$	4d N=4 Super Yang-Mills	✓ complete answer
11D	$AdS_4 \times S^7$	3d N=8 ABJM	⌛ partial answer
	$AdS_7 \times S^4$	6d N=(2,0)	⌛

Moreover, the bootstrap nature of these methods prevents us from looking under the hood, and understand the microscopic organizing principles of holographic correlators.

In this talk:

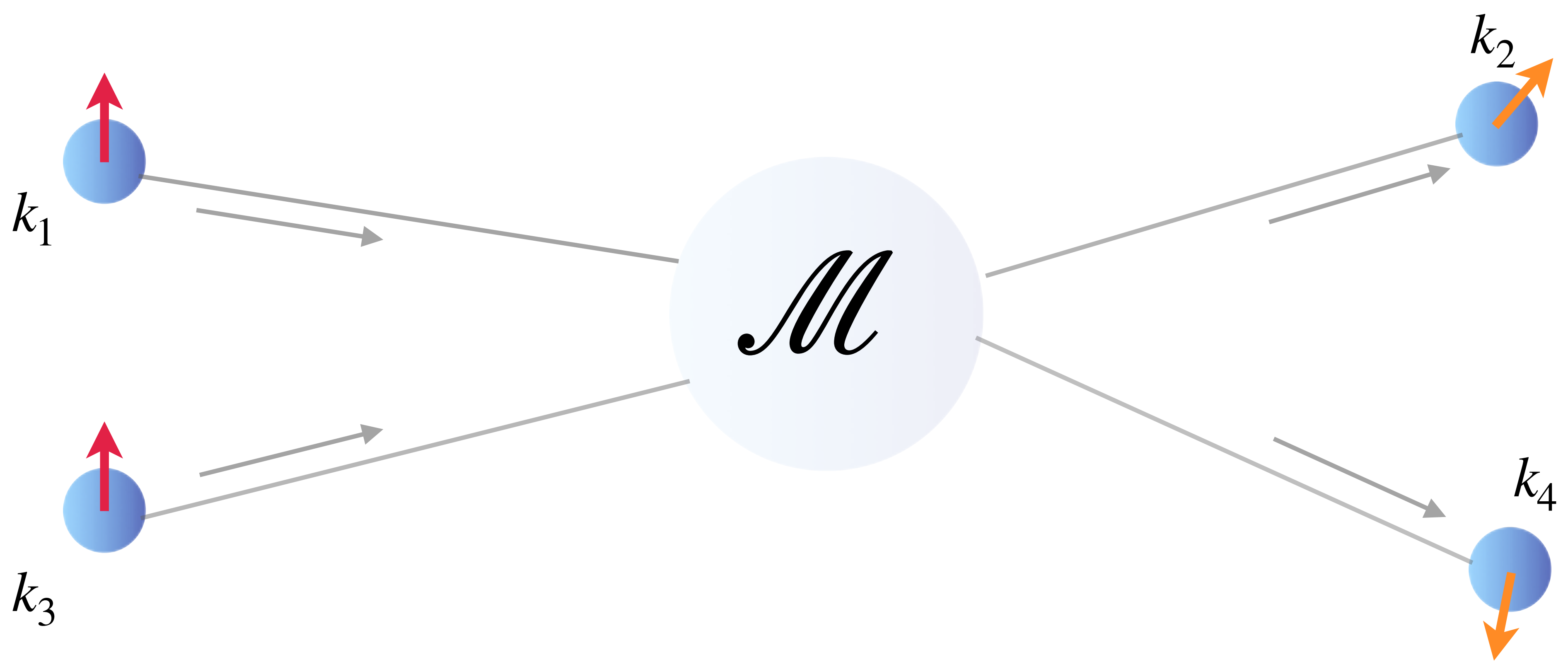
- Develop a *constructive* method to fully solve the other two more difficult cases $AdS_7 \times S^4$, $AdS_4 \times S^7$.
- Find the common organizing principles for amplitudes in different theories.

Introducing the MRV limit

- To achieve these goals, we should focus on the full amplitudes, not the reduced amplitudes.
- Reduced amplitudes can't be defined for $AdS_4 \times S^7$, and also don't have clear physical meanings.
- However, the full amplitude is very complicated.
- Perhaps we can first try to understand a simplifying limit? An idea from flat space: choose special polarizations!

Maximally ~~Helicity~~ Violating $\Rightarrow$ Maximally R-symmetry Violating

Recall the $(q+1)$-dimensional null vectors t_i^I for $SO(q+1)$ R-symmetry. We define MRV (in the u-channel) by setting $t_1^I = t_3^I$, or $\sigma = 0$, $\tau = 1$ in terms of the R-symmetry cross ratios.



$$\mathbf{MRV}(s,t) = \mathscr{M}(s,t;0,1)$$

Introducing the MRV limit

Let us look at the stress tensor multiplet four-point function in $AdS_7 \times S^4$.
The full amplitude seems like a big mess.

$$\mathcal{M}_{2222}(s,t;\sigma,\tau) = \frac{P_{22222}(s,t;\sigma,\tau)}{4n^3(s-4)(s-6)(t-4)(t-6)(u-4)(u-6)}$$

$$s+t+u=16$$

$$-5\,160\,960+2\,512\,896\,s-445\,440\,s^2+34\,176\,s^3-960\,s^4+4\,386\,816\,t-1\,967\,872\,s\,t+315\,008\,s^2\,t-$$
$$21\,392\,s^3\,t+520\,s^4\,t-1\,507\,072\,t^2+609\,152\,s\,t^2-84\,576\,s^2\,t^2+4764\,s^3\,t^2-90\,s^4\,t^2+$$
$$268\,800\,t^3-95\,264\,s\,t^3+10\,792\,s^2\,t^3-448\,s^3\,t^3+5\,s^4\,t^3-26\,320\,t^4+7892\,s\,t^4-654\,s^2\,t^4+$$
$$15\,s^3\,t^4+1344\,t^5-324\,s\,t^5+15\,s^2\,t^5-28\,t^6+5\,s\,t^6+3\,022\,848\,\sigma-2\,629\,632\,s\,\sigma+655\,872\,s^2\,\sigma-$$
$$61\,440\,s^3\,\sigma+1920\,s^4\,\sigma-2\,055\,168\,t\,\sigma+1\,797\,632\,s\,t\,\sigma-429\,760\,s^2\,t\,\sigma+37\,120\,s^3\,t\,\sigma-1040\,s^4\,t\,\sigma+$$
$$521\,984\,t^2\,\sigma-461\,760\,s\,t^2\,\sigma+103\,760\,s^2\,t^2\,\sigma-7840\,s^3\,t^2\,\sigma+180\,s^4\,t^2\,\sigma-61\,568\,t^3\,\sigma+$$
$$55\,760\,s\,t^3\,\sigma-11\,440\,s^2\,t^3\,\sigma+680\,s^3\,t^3\,\sigma-10\,s^4\,t^3\,\sigma+3328\,t^4\,\sigma-3168\,s\,t^4\,\sigma+568\,s^2\,t^4\,\sigma-$$
$$20\,s^3\,t^4\,\sigma-64\,t^5\,\sigma+68\,s\,t^5\,\sigma-10\,s^2\,t^5\,\sigma-1\,916\,928\,\sigma^2+1\,222\,656\,s\,\sigma^2-279\,552\,s^2\,\sigma^2+$$
$$27\,264\,s^3\,\sigma^2-960\,s^4\,\sigma^2+1\,222\,656\,t\,\sigma^2-762\,112\,s\,t\,\sigma^2+168\,704\,s^2\,t\,\sigma^2-15\,728\,s^3\,t\,\sigma^2+$$
$$520\,s^4\,t\,\sigma^2-279\,552\,t^2\,\sigma^2+168\,704\,s\,t^2\,\sigma^2-35\,568\,s^2\,t^2\,\sigma^2+3076\,s^3\,t^2\,\sigma^2-90\,s^4\,t^2\,\sigma^2+$$
$$27\,264\,t^3\,\sigma^2-15\,728\,s\,t^3\,\sigma^2+3076\,s^2\,t^3\,\sigma^2-232\,s^3\,t^3\,\sigma^2+5\,s^4\,t^3\,\sigma^2-960\,t^4\,\sigma^2+520\,s\,t^4\,\sigma^2-$$
$$90\,s^2\,t^4\,\sigma^2+5\,s^3\,t^4\,\sigma^2+2\,580\,480\,\tau-918\,528\,s\,\tau+57\,088\,s^2\,\tau+12\,416\,s^3\,\tau-1792\,s^4\,\tau+64\,s^5\,\tau-$$
$$918\,528\,t\,\tau-108\,544\,s\,t\,\tau+171\,200\,s^2\,t\,\tau-34\,256\,s^3\,t\,\tau+2592\,s^4\,t\,\tau-68\,s^5\,t\,\tau+57\,088\,t^2\,\tau+$$
$$171\,200\,s\,t^2\,\tau-74\,528\,s^2\,t^2\,\tau+10\,416\,s^3\,t^2\,\tau-572\,s^4\,t^2\,\tau+10\,s^5\,t^2\,\tau+12\,416\,t^3\,\tau-34\,256\,s\,t^3\,\tau+$$
$$10\,416\,s^2\,t^3\,\tau-1008\,s^3\,t^3\,\tau+30\,s^4\,t^3\,\tau-1792\,t^4\,\tau+2592\,s\,t^4\,\tau-572\,s^2\,t^4\,\tau+30\,s^3\,t^4\,\tau+$$
$$64\,t^5\,\tau-68\,s\,t^5\,\tau+10\,s^2\,t^5\,\tau+3\,022\,848\,\sigma\,\tau-2\,055\,168\,s\,\sigma\,\tau+521\,984\,s^2\,\sigma\,\tau-61\,568\,s^3\,\sigma\,\tau+$$
$$3328\,s^4\,\sigma\,\tau-64\,s^5\,\sigma\,\tau-2\,629\,632\,t\,\sigma\,\tau+1\,797\,632\,s\,t\,\sigma\,\tau-461\,760\,s^2\,t\,\sigma\,\tau+55\,760\,s^3\,t\,\sigma\,\tau-$$
$$3168\,s^4\,t\,\sigma\,\tau+68\,s^5\,t\,\sigma\,\tau+655\,872\,t^2\,\sigma\,\tau-429\,760\,s\,t^2\,\sigma\,\tau+103\,760\,s^2\,t^2\,\sigma\,\tau-11\,440\,s^3\,t^2\,\sigma\,\tau+$$
$$568\,s^4\,t^2\,\sigma\,\tau-10\,s^5\,t^2\,\sigma\,\tau-61\,440\,t^3\,\sigma\,\tau+37\,120\,s\,t^3\,\sigma\,\tau-7840\,s^2\,t^3\,\sigma\,\tau+680\,s^3\,t^3\,\sigma\,\tau-$$
$$20\,s^4\,t^3\,\sigma\,\tau+1920\,t^4\,\sigma\,\tau-1040\,s\,t^4\,\sigma\,\tau+180\,s^2\,t^4\,\sigma\,\tau-10\,s^3\,t^4\,\sigma\,\tau-5\,160\,960\,\tau^2+4\,386\,816\,s\,\tau^2-$$
$$1\,507\,072\,s^2\,\tau^2+268\,800\,s^3\,\tau^2-26\,320\,s^4\,\tau^2+1344\,s^5\,\tau^2-28\,s^6\,\tau^2+2\,512\,896\,t\,\tau^2-1\,967\,872\,s\,t\,\tau^2+$$
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$$4764\,s^2\,t^3\,\tau^2-448\,s^3\,t^3\,\tau^2+15\,s^4\,t^3\,\tau^2-960\,t^4\,\tau^2+520\,s\,t^4\,\tau^2-90\,s^2\,t^4\,\tau^2+5\,s^3\,t^4\,\tau^2$$

Introducing the MRV limit

Let us look at the stress tensor multiplet four-point function in $AdS_7 \times S^4$. The full amplitude seems like a big mess.

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$$s+t+u=16$$

$$\begin{aligned}
&-5160960 + 2512896\, s - 445440\, s^2 + 34176\, s^3 - 960\, s^4 + 4386816\, t - 1967872\, s\, t + 315008\, s^2\, t - \\
&21392\, s^3\, t + 520\, s^4\, t - 1507072\, t^2 + 609152\, s\, t^2 - 84576\, s^2\, t^2 + 4764\, s^3\, t^2 - 90\, s^4\, t^2 + \\
&268800\, t^3 - 95264\, s\, t^3 + 10792\, s^2\, t^3 - 448\, s^3\, t^3 + 5\, s^4\, t^3 - 26320\, t^4 + 7892\, s\, t^4 - 654\, s^2\, t^4 + \\
&15\, s^3\, t^4 + 1344\, t^5 - 324\, s\, t^5 + 15\, s^2\, t^5 - 28\, t^6 + 5\, s\, t^6 + 3022848\, \sigma - 2629632\, s\, \sigma + 655872\, s^2\, \sigma - \\
&61440\, s^3\, \sigma + 1920\, s^4\, \sigma - 2055168\, t\, \sigma + 1797632\, s\, t\, \sigma - 429760\, s^2\, t\, \sigma + 37120\, s^3\, t\, \sigma - 1040\, s^4\, t\, \sigma + \\
&521984\, t^2\, \sigma - 461760\, s\, t^2\, \sigma + 103760\, s^2\, t^2\, \sigma - 7840\, s^3\, t^2\, \sigma + 180\, s^4\, t^2\, \sigma - 61568\, t^3\, \sigma + \\
&55760\, s\, t^3\, \sigma - 11440\, s^2\, t^3\, \sigma + 680\, s^3\, t^3\, \sigma - 10\, s^4\, t^3\, \sigma + 3328\, t^4\, \sigma - 3168\, s\, t^4\, \sigma + 568\, s^2\, t^4\, \sigma - \\
&20\, s^3\, t^4\, \sigma - 64\, t^5\, \sigma + 68\, s\, t^5\, \sigma - 10\, s^2\, t^5\, \sigma - 1916928\, \sigma^2 + 1222656\, s\, \sigma^2 - 279552\, s^2\, \sigma^2 + \\
&27264\, s^3\, \sigma^2 - 960\, s^4\, \sigma^2 + 1222656\, t\, \sigma^2 - 762112\, s\, t\, \sigma^2 + 168704\, s^2\, t\, \sigma^2 - 15728\, s^3\, t\, \sigma^2 + \\
&520\, s^4\, t\, \sigma^2 - 279552\, t^2\, \sigma^2 + 168704\, s\, t^2\, \sigma^2 - 35568\, s^2\, t^2\, \sigma^2 + 3076\, s^3\, t^2\, \sigma^2 - 90\, s^4\, t^2\, \sigma^2 + \\
&27264\, t^3\, \sigma^2 - 15728\, s\, t^3\, \sigma^2 + 3076\, s^2\, t^3\, \sigma^2 - 232\, s^3\, t^3\, \sigma^2 + 5\, s^4\, t^3\, \sigma^2 - 960\, t^4\, \sigma^2 + 520\, s\, t^4\, \sigma^2 - \\
&90\, s^2\, t^4\, \sigma^2 + 5\, s^3\, t^4\, \sigma^2 + 2580480\, \tau - 918528\, s\, \tau + 57088\, s^2\, \tau + 12416\, s^3\, \tau - 1792\, s^4\, \tau + 64\, s^5\, \tau - \\
&918528\, t\, \tau - 108544\, s\, t\, \tau + 171200\, s^2\, t\, \tau - 34256\, s^3\, t\, \tau + 2592\, s^4\, t\, \tau - 68\, s^5\, t\, \tau + 57088\, t^2\, \tau + \\
&171200\, s\, t^2\, \tau - 74528\, s^2\, t^2\, \tau + 10416\, s^3\, t^2\, \tau - 572\, s^4\, t^2\, \tau + 10\, s^5\, t^2\, \tau + 12416\, t^3\, \tau - 34256\, s\, t^3\, \tau + \\
&10416\, s^2\, t^3\, \tau - 1008\, s^3\, t^3\, \tau + 30\, s^4\, t^3\, \tau - 1792\, t^4\, \tau + 2592\, s\, t^4\, \tau - 572\, s^2\, t^4\, \tau + 30\, s^3\, t^4\, \tau + \\
&64\, t^5\, \tau - 68\, s\, t^5\, \tau + 10\, s^2\, t^5\, \tau + 3022848\, \sigma\, \tau - 2055168\, s\, \sigma\, \tau + 521984\, s^2\, \sigma\, \tau - 61568\, s^3\, \sigma\, \tau + \\
&3328\, s^4\, \sigma\, \tau - 64\, s^5\, \sigma\, \tau - 2629632\, t\, \sigma\, \tau + 1797632\, s\, t\, \sigma\, \tau - 461760\, s^2\, t\, \sigma\, \tau + 55760\, s^3\, t\, \sigma\, \tau - \\
&3168\, s^4\, t\, \sigma\, \tau + 68\, s^5\, t\, \sigma\, \tau + 655872\, t^2\, \sigma\, \tau - 429760\, s\, t^2\, \sigma\, \tau + 103760\, s^2\, t^2\, \sigma\, \tau - 11440\, s^3\, t^2\, \sigma\, \tau + \\
&568\, s^4\, t^2\, \sigma\, \tau - 10\, s^5\, t^2\, \sigma\, \tau - 61440\, t^3\, \sigma\, \tau + 37120\, s\, t^3\, \sigma\, \tau - 7840\, s^2\, t^3\, \sigma\, \tau + 680\, s^3\, t^3\, \sigma\, \tau - \\
&20\, s^4\, t^3\, \sigma\, \tau + 1920\, t^4\, \sigma\, \tau - 1040\, s\, t^4\, \sigma\, \tau + 180\, s^2\, t^4\, \sigma\, \tau - 10\, s^3\, t^4\, \sigma\, \tau - 5160960\, \tau^2 + 4386816\, s\, \tau^2 - \\
&1507072\, s^2\, \tau^2 + 268800\, s^3\, \tau^2 - 26320\, s^4\, \tau^2 + 1344\, s^5\, \tau^2 - 28\, s^6\, \tau^2 + 2512896\, t\, \tau^2 - 1967872\, s\, t\, \tau^2 + \\
&609152\, s^2\, t\, \tau^2 - 95264\, s^3\, t\, \tau^2 + 7892\, s^4\, t\, \tau^2 - 324\, s^5\, t\, \tau^2 + 5\, s^6\, t\, \tau^2 - 445440\, t^2\, \tau^2 + 315008\, s\, t^2\, \tau^2 - \\
&84576\, s^2\, t^2\, \tau^2 + 10792\, s^3\, t^2\, \tau^2 - 654\, s^4\, t^2\, \tau^2 + 15\, s^5\, t^2\, \tau^2 + 34176\, t^3\, \tau^2 - 21392\, s\, t^3\, \tau^2 + \\
&4764\, s^2\, t^3\, \tau^2 - 448\, s^3\, t^3\, \tau^2 + 15\, s^4\, t^3\, \tau^2 - 960\, t^4\, \tau^2 + 520\, s\, t^4\, \tau^2 - 90\, s^2\, t^4\, \tau^2 + 5\, s^3\, t^4\, \tau^2
\end{aligned}$$

However, big simplification in the MRV limit $\sigma = 0, \tau = 1$.

$$\mathbf{MRV}_{2222}(s,t) = \frac{(u-8)(u-10)}{n^3}\left(\frac{1}{s-4} + \frac{1}{4(s-6)} + \frac{1}{t-4} + \frac{1}{4(t-6)}\right)$$

Introducing the MRV limit

Let us look at the stress tensor multiplet four-point function in $AdS_7 \times S^4$.
The full amplitude seems like a big mess.

$$\mathcal{M}_{2222}(s,t;\sigma,\tau) = \frac{P_{22222}(s,t;\sigma,\tau)}{4n^3(s-4)(s-6)(t-4)(t-6)(u-4)(u-6)}$$

$$\begin{aligned}
&-5\,160\,960 + 2\,512\,896\, s - 445\,440\, s^2 + 34\,176\, s^3 - 960\, s^4 + 4\,386\,816\, t - 1\,967\,872\, s\, t + 315\,008\, s^2\, t - \\
&21\,392\, s^3\, t + 520\, s^4\, t - 1\,507\,072\, t^2 + 609\,152\, s\, t^2 - 84\,576\, s^2\, t^2 + 4764\, s^3\, t^2 - 90\, s^4\, t^2 + \\
&268\,800\, t^3 - 95\,264\, s\, t^3 + 10\,792\, s^2\, t^3 - 448\, s^3\, t^3 + 5\, s^4\, t^3 - 26\,320\, t^4 + 7892\, s\, t^4 - 654\, s^2\, t^4 + \\
&15\, s^3\, t^4 + 1344\, t^5 - 324\, s\, t^5 + 15\, s^2\, t^5 - 28\, t^6 + 5\, s\, t^6 + 3\,022\,848\, \sigma - 2\,629\,632\, s\, \sigma + 655\,872\, s^2\, \sigma - \\
&61\,440\, s^3\, \sigma + 1920\, s^4\, \sigma - 2\,055\,168\, t\, \sigma + 1\,797\,632\, s\, t\, \sigma - 429\,760\, s^2\, t\, \sigma + 37\,120\, s^3\, t\, \sigma - 1040\, s^4\, t\, \sigma + \\
&521\,984\, t^2\, \sigma - 461\,760\, s\, t^2\, \sigma + 103\,760\, s^2\, t^2\, \sigma - 7840\, s^3\, t^2\, \sigma + 180\, s^4\, t^2\, \sigma - 61\,568\, t^3\, \sigma + \\
&55\,760\, s\, t^3\, \sigma - 11\,440\, s^2\, t^3\, \sigma + 680\, s^3\, t^3\, \sigma - 10\, s^4\, t^3\, \sigma + 3328\, t^4\, \sigma - 3168\, s\, t^4\, \sigma + 568\, s^2\, t^4\, \sigma - \\
&20\, s^3\, t^4\, \sigma - 64\, t^5\, \sigma + 68\, s\, t^5\, \sigma - 10\, s^2\, t^5\, \sigma - 1\,916\,928\, \sigma^2 + 1\,222\,656\, s\, \sigma^2 - 279\,552\, s^2\, \sigma^2 + \\
&27\,264\, s^3\, \sigma^2 - 960\, s^4\, \sigma^2 + 1\,222\,656\, t\, \sigma^2 - 762\,112\, s\, t\, \sigma^2 + 168\,704\, s^2\, t\, \sigma^2 - 15\,728\, s^3\, t\, \sigma^2 + \\
&520\, s^4\, t\, \sigma^2 - 279\,552\, t^2\, \sigma^2 + 168\,704\, s\, t^2\, \sigma^2 - 35\,568\, s^2\, t^2\, \sigma^2 + 3076\, s^3\, t^2\, \sigma^2 - 90\, s^4\, t^2\, \sigma^2 + \\
&27\,264\, t^3\, \sigma^2 - 15\,728\, s\, t^3\, \sigma^2 + 3076\, s^2\, t^3\, \sigma^2 - 232\, s^3\, t^3\, \sigma^2 + 5\, s^4\, t^3\, \sigma^2 - 960\, t^4\, \sigma^2 + 520\, s\, t^4\, \sigma^2 - \\
&90\, s^2\, t^4\, \sigma^2 + 5\, s^3\, t^4\, \sigma^2 + 2\,580\,480\, \tau - 918\,528\, s\, \tau + 57\,088\, s^2\, \tau + 12\,416\, s^3\, \tau - 1792\, s^4\, \tau + 64\, s^5\, \tau - \\
&918\,528\, t\, \tau - 108\,544\, s\, t\, \tau + 171\,200\, s^2\, t\, \tau - 34\,256\, s^3\, t\, \tau + 2592\, s^4\, t\, \tau - 68\, s^5\, t\, \tau + 57\,088\, t^2\, \tau + \\
&171\,200\, s\, t^2\, \tau - 74\,528\, s^2\, t^2\, \tau + 10\,416\, s^3\, t^2\, \tau - 572\, s^4\, t^2\, \tau + 10\, s^5\, t^2\, \tau + 12\,416\, t^3\, \tau - 34\,256\, s\, t^3\, \tau + \\
&10\,416\, s^2\, t^3\, \tau - 1008\, s^3\, t^3\, \tau + 30\, s^4\, t^3\, \tau - 1792\, t^4\, \tau + 2592\, s\, t^4\, \tau - 572\, s^2\, t^4\, \tau + 30\, s^3\, t^4\, \tau + \\
&64\, t^5\, \tau - 68\, s\, t^5\, \tau + 10\, s^2\, t^5\, \tau + 3\,022\,848\, \sigma\tau - 2\,055\,168\, s\, \sigma\tau + 521\,984\, s^2\, \sigma\tau - 61\,568\, s^3\, \sigma\tau + \\
&3328\, s^4\, \sigma\tau - 64\, s^5\, \sigma\tau - 2\,629\,632\, t\, \sigma\tau + 1\,797\,632\, s\, t\, \sigma\tau - 461\,760\, s^2\, t\, \sigma\tau + 55\,760\, s^3\, t\, \sigma\tau - \\
&3168\, s^4\, t\, \sigma\tau + 68\, s^5\, t\, \sigma\tau + 655\,872\, t^2\, \sigma\tau - 429\,760\, s\, t^2\, \sigma\tau + 103\,760\, s^2\, t^2\, \sigma\tau - 11\,440\, s^3\, t^2\, \sigma\tau + \\
&568\, s^4\, t^2\, \sigma\tau - 10\, s^5\, t^2\, \sigma\tau - 61\,440\, t^3\, \sigma\tau + 37\,120\, s\, t^3\, \sigma\tau - 7840\, s^2\, t^3\, \sigma\tau + 680\, s^3\, t^3\, \sigma\tau - \\
&20\, s^4\, t^3\, \sigma\tau + 1920\, t^4\, \sigma\tau - 1040\, s\, t^4\, \sigma\tau + 180\, s^2\, t^4\, \sigma\tau - 10\, s^3\, t^4\, \sigma\tau - 5\,160\,960\, \tau^2 + 4\,386\,816\, s\, \tau^2 - \\
&1\,507\,072\, s^2\, \tau^2 + 268\,800\, s^3\, \tau^2 - 26\,320\, s^4\, \tau^2 + 1344\, s^5\, \tau^2 - 28\, s^6\, \tau^2 + 2\,512\,896\, t\, \tau^2 - 1\,967\,872\, s\, t\, \tau^2 + \\
&609\,152\, s^2\, t\, \tau^2 - 95\,264\, s^3\, t\, \tau^2 + 7892\, s^4\, t\, \tau^2 - 324\, s^5\, t\, \tau^2 + 5\, s^6\, t\, \tau^2 - 445\,440\, t^2\, \tau^2 + 315\,008\, s\, t^2\, \tau^2 - \\
&84\,576\, s^2\, t^2\, \tau^2 + 10\,792\, s^3\, t^2\, \tau^2 - 654\, s^4\, t^2\, \tau^2 + 15\, s^5\, t^2\, \tau^2 + 34\,176\, t^3\, \tau^2 - 21\,392\, s\, t^3\, \tau^2 + \\
&4764\, s^2\, t^3\, \tau^2 - 448\, s^3\, t^3\, \tau^2 + 15\, s^4\, t^3\, \tau^2 - 960\, t^4\, \tau^2 + 520\, s\, t^4\, \tau^2 - 90\, s^2\, t^4\, \tau^2 + 5\, s^3\, t^4\, \tau^2
\end{aligned}$$

$$s + t + u = 16$$

However, big simplification in the MRV limit $\sigma = 0, \tau = 1$.

$$\mathbf{MRV}_{2222}(s,t) = \frac{(u-8)(u-10)}{n^3}\left(\frac{1}{s-4} + \frac{1}{4(s-6)} + \frac{1}{t-4} + \frac{1}{4(t-6)}\right)$$

Zeroes at the double-trace locations

No poles in the u-channel

Properties of MRV amplitudes

Let us try to understand the physics behind these two features.

$$\mathbf{MRV}_{2222}(s,t) = \frac{(u-8)(u-10)}{n^3}\left(\frac{1}{s-4} + \frac{1}{4(s-6)} + \frac{1}{t-4} + \frac{1}{4(t-6)}\right)$$

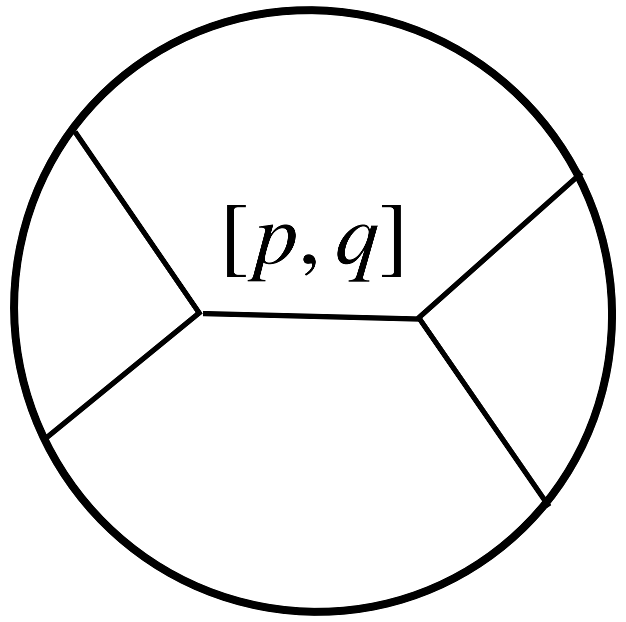
No u-channel poles: Each supergravity field has an irrep under R-symmetry. This information is captured by a R-symmetry polynomial of σ, τ.

$$\mathcal{Y}^{(channel)}_{[p,q]}(\sigma,\tau)$$

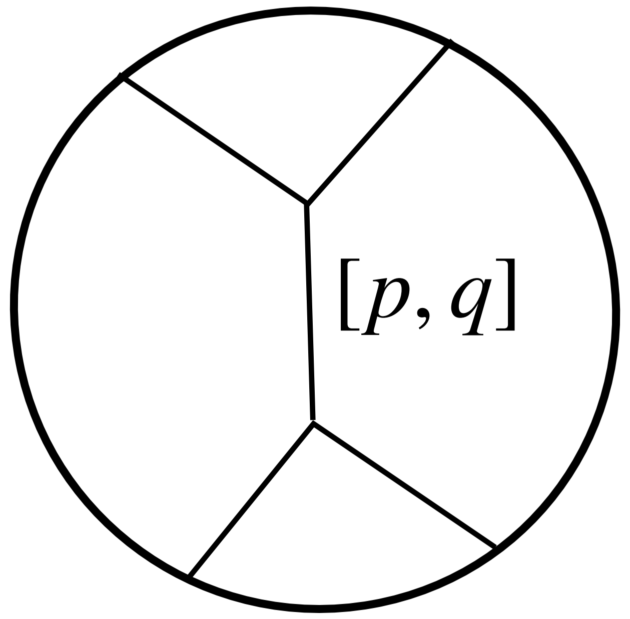
field	s_p	$A_{p,\mu}$	$\varphi_{p,\mu\nu}$	$C_{p,\mu}$	t_p	r_p
ℓ	0	1	2	1	0	0
Δ	$2p$	$2p+1$	$2p+2$	$2p+3$	$2p+4$	$2p+2$
SO(5)	$[p,0]$	$[p-2,2]$	$[p-2,0]$	$[p-4,2]$	$[p-4,0]$	$[p-4,4]$

superconformal primary

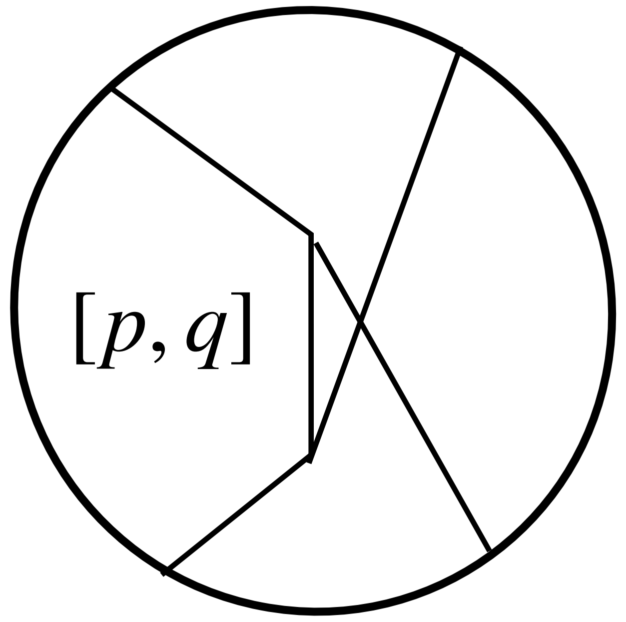
Eigenfunction of the 2-particle quadratic Casimir for R-symmetry. The u-channel polynomials vanish in the MRV limit.



$\propto \mathcal{Y}^{(s)}_{[p,q]}(\sigma,\tau)$



$\propto \mathcal{Y}^{(t)}_{[p,q]}(\sigma,\tau)$



zero in the MRV limit

$\propto \mathcal{Y}^{(u)}_{[p,q]}(\sigma,\tau)$

To say it differently, $t_1 = t_3$ means only rank-$(k_1 + k_3)$ symmetric traceless representation can appear in the u-channel. The s- and t-channel representations have an overlap, but the u-channel exchange fields do not.

Properties of MRV amplitudes

Let us try to understand the physics behind these two features.

Zeroes in u: **1.** low-twist double-trace long operators decouple in the MRV configuration. **2.** The location of the zeroes correspond to the conformal twists of the long operators.

The double poles from the Gamma factor

$$\int \frac{dsdt}{(4\pi i)^2} U^{\frac{s}{2}} V^{\frac{t}{2}-4} \mathcal{M}(s,t) \Gamma^2[\frac{8-s}{2}] \Gamma^2[\frac{8-t}{2}] \Gamma^2[\frac{8-u}{2}]$$

gives logs in cross ratios upon taking residues, which are associated with anomalous dimensions — hallmarks of long operators. To decouple, we need zeroes.

More generally, the two zeroes are at

$$u = 2\max\{k_1+k_3, k_2+k_4\}, 2\max\{k_1+k_3, k_2+k_4\}+2$$

$$\mathbf{MRV}_{2222}(s,t) = \frac{(u-8)(u-10)}{n^3}\left(\frac{1}{s-4} + \frac{1}{4(s-6)} + \frac{1}{t-4} + \frac{1}{4(t-6)}\right)$$

[0 0]	[0 2]	[2 0]	[0 4]	[2 2]	[4 0]
$(\Delta)_\ell$	$(\Delta+1)_{\ell-1}$	$(\Delta+2)_{\ell-2}$	$(\Delta+2)_\ell$	$(\Delta+3)_{\ell-1}$	$(\Delta+4)_\ell$
$(\Delta+2)_{\ell-2}$	$(\Delta+1)_{\ell+1}$	$(\Delta+2)_\ell$	$(\Delta+4)_{\ell-2}$	$(\Delta+3)_{\ell+1}$	
$(\Delta+2)_\ell$	$(\Delta+3)_{\ell-3}$	$(\Delta+2)_{\ell+2}$	$(\Delta+4)_\ell$	$(\Delta+5)_{\ell-1}$	
$(\Delta+2)_{\ell+2}$	$(\Delta+3)_{\ell-1}$	$(\Delta+4)_{\ell-2}$	$(\Delta+4)_{\ell+2}$	$(\Delta+5)_{\ell+1}$	
$(\Delta+4)_{\ell-4}$	$(\Delta+3)_{\ell+1}$	$(\Delta+4)_\ell$	$(\Delta+6)_\ell$		
$(\Delta+4)_{\ell-2}$	$(\Delta+3)_{\ell+3}$	$(\Delta+4)_{\ell+2}$			
$(\Delta+4)_\ell$	$(\Delta+5)_{\ell-3}$	$(\Delta+6)_{\ell-2}$			
$(\Delta+4)_{\ell+2}$	$(\Delta+5)_{\ell-1}$	$(\Delta+6)_\ell$			
$(\Delta+4)_{\ell+4}$	$(\Delta+5)_{\ell+1}$	$(\Delta+6)_{\ell+2}$			
$(\Delta+6)_{\ell-2}$	$(\Delta+5)_{\ell+3}$				
$(\Delta+6)_\ell$	$(\Delta+7)_{\ell-1}$				
$(\Delta+6)_{\ell+2}$	$(\Delta+7)_{\ell+1}$				
$(\Delta+8)_\ell$					

[Dolan, Osborn; Beem, Lemos, Rastelli, van Rees]

In $\langle 2222 \rangle$ only [4,0] survives the MRV limit. There are no double-trace operators (super primary or descendant) in the u-channel with twists $\Delta-\ell$, $\Delta-\ell+2$. Moreover, all the long operators at the supergravity limit are double-trace operators. This means $\Delta-\ell \geq 8$, and the first long operator has twist $\tau=12$.

Computing MRV amplitudes

The zeros in fact are satisfied by each multiplet in each channel.

$\Rightarrow$ nontrivial constraints to fix the *relative coefficients* of *all* the component fields in the same multiplet $\Rightarrow$ obtain MRV amplitudes.

field	s_p	$A_{p,\mu}$	$\varphi_{p,\mu\nu}$	$C_{p,\mu}$	t_p	r_p
ℓ	0	1	2	1	0	0
Δ	$2p$	$2p+1$	$2p+2$	$2p+3$	$2p+4$	$2p+2$
SO(5)	$[p,0]$	$[p-2,2]$	$[p-2,0]$	$[p-4,2]$	$[p-4,0]$	$[p-4,4]$

1. Ansatz

$$\mathcal{S}^{(s)}_p = \lambda_s \mathcal{Y}_{\{p,0\}} \mathcal{M}^{(s)}_{2p,0} + \lambda_A \mathcal{Y}_{\{p-2,2\}} \mathcal{M}^{(s)}_{2p+1,1} + \lambda_\varphi \mathcal{Y}_{\{p-2,0\}} \mathcal{M}^{(s)}_{2p+2,2}$$
$$+\lambda_C \mathcal{Y}_{\{p-4,2\}} \mathcal{M}^{(s)}_{2p+3,1} + \lambda_t \mathcal{Y}_{\{p-4,0\}} \mathcal{M}^{(s)}_{2p+4,0} + \lambda_r \mathcal{Y}_{\{p-4,4\}} \mathcal{M}^{(s)}_{2p+2,0}$$

2. Use the *"Polyakov-Regge blocks"* [Mazac, Rastelli, XZ; Sleight Taronna]

$$\mathcal{M}^{(s)}_{\Delta,\ell} \to P^{(s)}_{\Delta,\ell}$$

corresponds to a specific choice of contact terms, to match the improved Regge behavior of the MRV amplitude.

3. Impose zeroes and solve all λ_{field} in terms of λ_s, which are known are super primary 3-pt functions.

All MRV amplitudes solved as a sum over multiplets! (by selection rules, there are only finitely many multiplets!)

The u-channel Regge limit is to take $s \to \infty$, keeping u fixed. In this limit, the amplitude

$$\mathbf{MRV}^{(s)}_{2222}(s,t) = \frac{(u-8)(u-10)}{n^3}\left(\frac{1}{s-4} + \frac{1}{4(s-6)}\right) \sim 1/s$$

while a spin-ℓ exchange with generic choice of cubic vertices grows like $s^{\ell-1}$.

$$\mathcal{M}^{(s)}_{\Delta,\ell} = \sum_{m=0}^{\infty} \frac{\mathcal{Q}_{m,\ell}(\cancel{t},u)}{s-(\Delta-\ell)-2m} + \cancel{P_{\ell-1}(s,t)}$$

$$P^{(s)}_{\Delta,\ell}(s,u) = \sum_{m=0}^{\infty} \frac{\mathcal{Q}_{m,\ell}(\epsilon\Sigma - u - (\Delta-\ell) - 2m, u)}{s-\Delta+\ell-2m}$$

Full amplitudes from MRV amplitudes

Solving all MRV amplitudes is nice. But we can also obtain the *full amplitude* from this limit!

1. Restore the full σ, τ dependence in the exchange contributions

$$\widetilde{\mathcal{S}}_p^{(s)} = \lambda_s \mathcal{Y}_{\{p,0\}}(\sigma,\tau)P^{(s)}_{2p,0} + \lambda_A \mathcal{Y}_{\{p-2,2\}}(\sigma,\tau)P^{(s)}_{2p+1,1} + \lambda_\varphi \mathcal{Y}_{\{p-2,0\}}(\sigma,\tau)P^{(s)}_{2p+2,2}$$

$$+\lambda_C \mathcal{Y}_{\{p-4,2\}}(\sigma,\tau)P^{(s)}_{2p+3,1} + \lambda_t \mathcal{Y}_{\{p-4,0\}}(\sigma,\tau)P^{(s)}_{2p+4,0} + \lambda_r \mathcal{Y}_{\{p-4,4\}}(\sigma,\tau)P^{(s)}_{2p+2,0}$$

2. Polar part fixed. Regular terms? They are *uniquely* fixed by superconformal Ward identities in Mellin space! Note however we can move contact into exchange, so the division is not absolute.

3. Remarkably, we found a prescription for completing the MRV multiplet exchange amplitudes into the *full amplitude*, so that *no explicit contact terms* are needed! This prescription is related to the restoration of Bose symmetry:

   - The multiplet exchange $\widetilde{\mathcal{S}}_p^{(s)}$ amplitudes do not have Bose symmetry in 1, 2 and 3, 4.

   - To restore Bose symmetry, we notice that at each pole it has a factor

$$\left(u^2 + \alpha(i,j;m,p)u + \beta(i,j;m,p)\right)\sigma^i\tau^j$$

We simply use $s+t+u = 2(k_1+k_2+k_3+k_4)$, and that the poles are at $s = 2p+2m$ to trade m for t. The Bose symmetry is restored. Meanwhile, the improved u-channel Regge behavior in the MRV limit is still preserved.

Full amplitudes from MRV amplitudes

Similar constructions for $\widetilde{\mathcal{S}}_p^{(t)}$, $\widetilde{\mathcal{S}}_p^{(u)}$, and they are related to $\widetilde{\mathcal{S}}_p^{(s)}$ by crossing.

Upshot: full answer is the sum over exchange contributions from multiplets p. The multiplet p is from a finite range

$$p - \max\{|k_1 - k_2|, |k_3 - k_4|\} = 2,4,\ldots 2\mathcal{E} - 2$$

constrained by selection rules (R-symmetry & vanishing of extremal couplings).

No additional contact terms!

Here we demonstrated our construction with $AdS_7 \times S^4$, but it is exactly the same for $AdS_5 \times S^5$ and $AdS_4 \times S^7$. They have the same structure of multiplets, and also contact terms can be shown to be absent.

field	s_p	$A_{p,\mu}$	$\varphi_{p,\mu\nu}$	$C_{p,\mu}$	t_p	r_p
ℓ	0	1	2	1	0	0
Δ	ϵp	$\epsilon p+1$	$\epsilon p+2$	$\epsilon p+3$	$\epsilon p+4$	$\epsilon p+2$
d_1	p	$p-2$	$p-2$	$p-4$	$p-4$	$p-4$
d_2	0	2	0	2	0	4

Here recall $\epsilon = \frac{d-2}{2}$. $\{d_1, d_2\}$ are R-symmetry quantum numbers

$SO(5)$: $[d_1, d_2]$, $SU(4)$: $[\frac{d_2}{2}, d_1, \frac{d_2}{2}]$, $SO(8)$: $[d_1, \frac{d_2}{2}, 0, 0]$

All four-point Mellin amplitudes

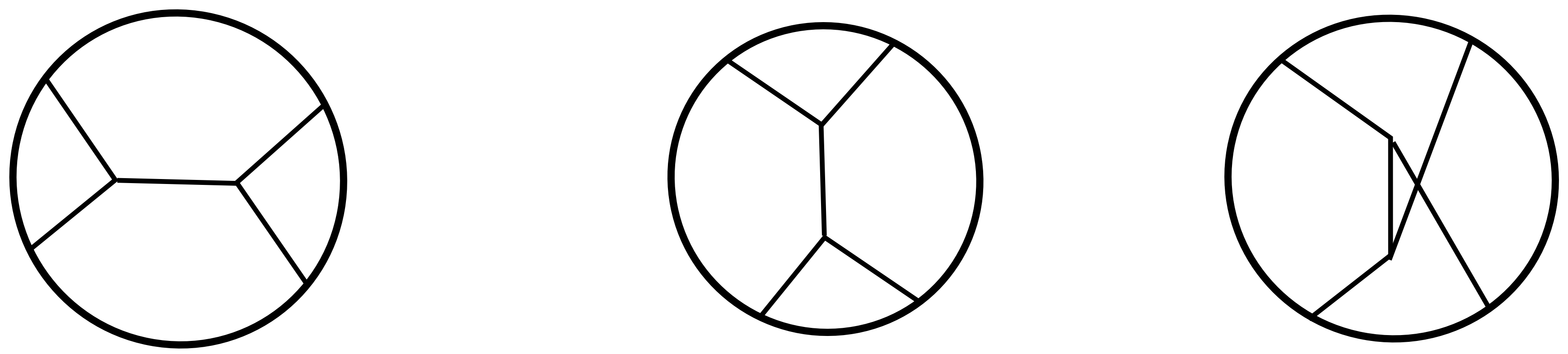
To summarize:

MRV limit (drastic simplification & easy to construct) —using symmetries→ full amplitudes

The amplitudes have the following structure:

$$\mathcal{M}(s,t;\sigma,\tau) = \mathcal{M}_s(s,t;\sigma,\tau) + \mathcal{M}_t(s,t;\sigma,\tau) + \mathcal{M}_u(s,t;\sigma,\tau)$$

(no contact contribution)



$$\mathcal{M}_s(s,t;\sigma,\tau) = \sum_{i,j} \sigma^i \tau^j \sum_{s_0} \frac{R_s^{i,j}(t,u)}{s-s_0}, \qquad s_0 = \epsilon p + 2m\ , \quad m \in \mathbb{N}$$

(sum over simple poles)

$$R_{s_0}^{i,j}(t,u) = \sum_p \mathcal{R}_{p,m}^{i,j}(t,u) \qquad p - \max\{\,|k_1-k_2|\,,|k_3-k_4|\,\} = 2,4,\ldots 2\mathcal{E}-2$$

(finite sum over multiplets)

The residues have the following *universal* structure

$$\mathcal{R}_{p,m}^{i,j}(t,u) = K_p^{i,j}(t,u)\, L_{p,m}^{i,j}\, N_p^{i,j},$$

All four-point Mellin amplitudes

Final results for all three backgrounds:

$$\mathscr{R}^{i,j}_{p,m}(t,u) = K^{i,j}_p(t,u)\, L^{i,j}_{p,m}\, N^{i,j}_p$$

$$K^{i,j}_p = 2i(2i+\kappa_u)t^-t^+ + 2j(2j+\kappa_t)u^-u^+ - 2j(\tfrac{2}{\epsilon}-2+\kappa_u)t^+u^- - 2i(\tfrac{2}{\epsilon}-2+\kappa_t)u^+t^-$$
$$+\frac{1}{4}(2p-\kappa_t-\kappa_u)(2p+\tfrac{4}{\epsilon}-4+\kappa_t+\kappa_u)(u^-t^-+4\epsilon^2 ij)$$
$$+\tfrac{\epsilon}{2}(\kappa_u+\kappa_t-2p)(\kappa_u+\kappa_t+2p+\tfrac{4}{\epsilon}-4)(it^-+ju^-)$$
$$+4\epsilon ij(t^+(\kappa_u+\tfrac{2}{\epsilon}-2)+u^+(\kappa_t+\tfrac{2}{\epsilon}-2)) - 8ijt^+u^+\,,$$

$$L^{i,j}_{p,m} = \frac{\pi^{-\frac{(\epsilon-1)(2\epsilon+5)}{6}}\, 2^{\frac{2(\epsilon-1)(2\epsilon-1)}{3}} \prod_{i=1}^4 \left(\sqrt{k_i+\frac{1}{\epsilon}-1}\,\Gamma[\frac{2}{3}((1+\epsilon)k_i+2-\epsilon)]^{\frac{1}{3}(\frac{1}{\epsilon}-\epsilon)}\right)}{n^{1+\epsilon}\,\Gamma[2-\epsilon+m+\epsilon p]}$$
$$\times\frac{\left(\Gamma[\frac{(1+\epsilon)(k_1+k_2+p)}{6}+\frac{2(2-\epsilon)}{3}]\Gamma[\frac{(1+\epsilon)(k_3+k_4+p)}{6}+\frac{2(2-\epsilon)}{3}]\right)^{-\frac{2}{3}(\frac{1}{\epsilon}-\epsilon)}}{i!\,j!\,m!}$$
$$\times\frac{(-1)^{i+j+\frac{2p-\kappa_t-\kappa_u}{4}}\left(\Gamma[\frac{(1+\epsilon)(k_1+k_2-p)}{6}+\frac{1}{2}]\Gamma[\frac{(1+\epsilon)(k_3+k_4-p)}{6}+\frac{1}{2}]\right)^{-\frac{2}{3}(\frac{1}{\epsilon}-\epsilon)}}{\Gamma[\frac{\epsilon}{2}(k_1+k_2-p)-m]\Gamma[\frac{\epsilon}{2}(k_3+k_4-p)-m]}\,,$$

$$\epsilon = \frac{d-2}{2}$$

$$N^{i,j}_p = \frac{2^{\Sigma(\epsilon-1)-4-\epsilon}\Gamma[\frac{1}{4}(\frac{4}{\epsilon}-4+2p+\Sigma-\kappa_s-4l)](-(\frac{5\epsilon^2-15\epsilon+6}{\epsilon})p+1-\epsilon)}{\Gamma[\frac{\kappa_u+2+2i}{2}]\Gamma[\frac{2(p+2)-\Sigma+\kappa_s+4l}{4}]\Gamma[\frac{\kappa_t+2+2j}{2}]}\,.$$

Definitions:

$$u^\pm = u \pm \frac{\epsilon}{2}\kappa_u - \frac{\epsilon}{2}\Sigma\,,\qquad t^\pm = t\pm\frac{\epsilon}{2}\kappa_t - \frac{\epsilon}{2}\Sigma$$

$$\kappa_s \equiv |k_3+k_4-k_1-k_2|\,,\quad \kappa_t \equiv |k_1+k_4-k_2-k_3|\,,$$

$$\kappa_u \equiv |k_2+k_4-k_1-k_3|$$

$$\Sigma = k_1+k_2+k_3+k_4 \qquad i+j+l=\mathcal{E}$$

There are three physical values for ϵ.

$\epsilon = 1$: reproduces previous results in $AdS_5\times S^5$,

$\epsilon = 2$: new results for all correlators in $AdS_7\times S^4$,

$\epsilon = \frac{1}{2}$: new results for all correlators in $AdS_4\times S^7$.

These expressions show a universality for amplitudes from different theories.

Hidden structures

The results turn out to have amazing hidden structures. The amplitudes can be massaged such that a lower dimensional spacetime naturally appears. This emergent dimensional reduction is closely related to the Parisi-Sourlas supersymmetry that was first found in statistical mechanics problems.

1. Let's first focus on the AdS part. We can extract a differential operator from each multiplet exchange amplitude.

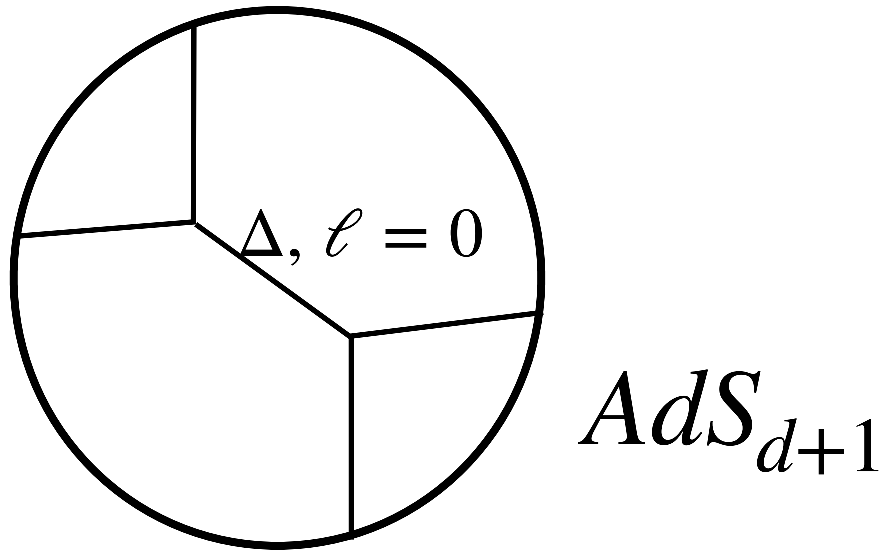
$$\mathscr{R}^{i,j}_{p,m}(t,u) = K^{i,j}_{p}(t,u)\, L^{i,j}_{p,m}\, N^{i,j}_{p}$$

$K^{i,j}_p(t,u)$ is a quadratic polynomial in t, u, so we can move it out of the inverse Mellin integrals as differential operators: $y\partial_y \int dt\, y^t(\ldots) = \int dt\, y^t t \times (\ldots)$.

2. The remaining dependence on Mandelstam variables surprisingly can be written as the sum of three scalar exchange Witten diagrams in AdS_{d+1}, with shifted internal conformal dimensions

$$\text{diff operator} \circ (M^{(s),d}_{\epsilon p} + \#\, M^{(s),d}_{\epsilon p+2} + \#\, M^{(s),d}_{\epsilon p+4})$$

Similar to weight-shifting operators!

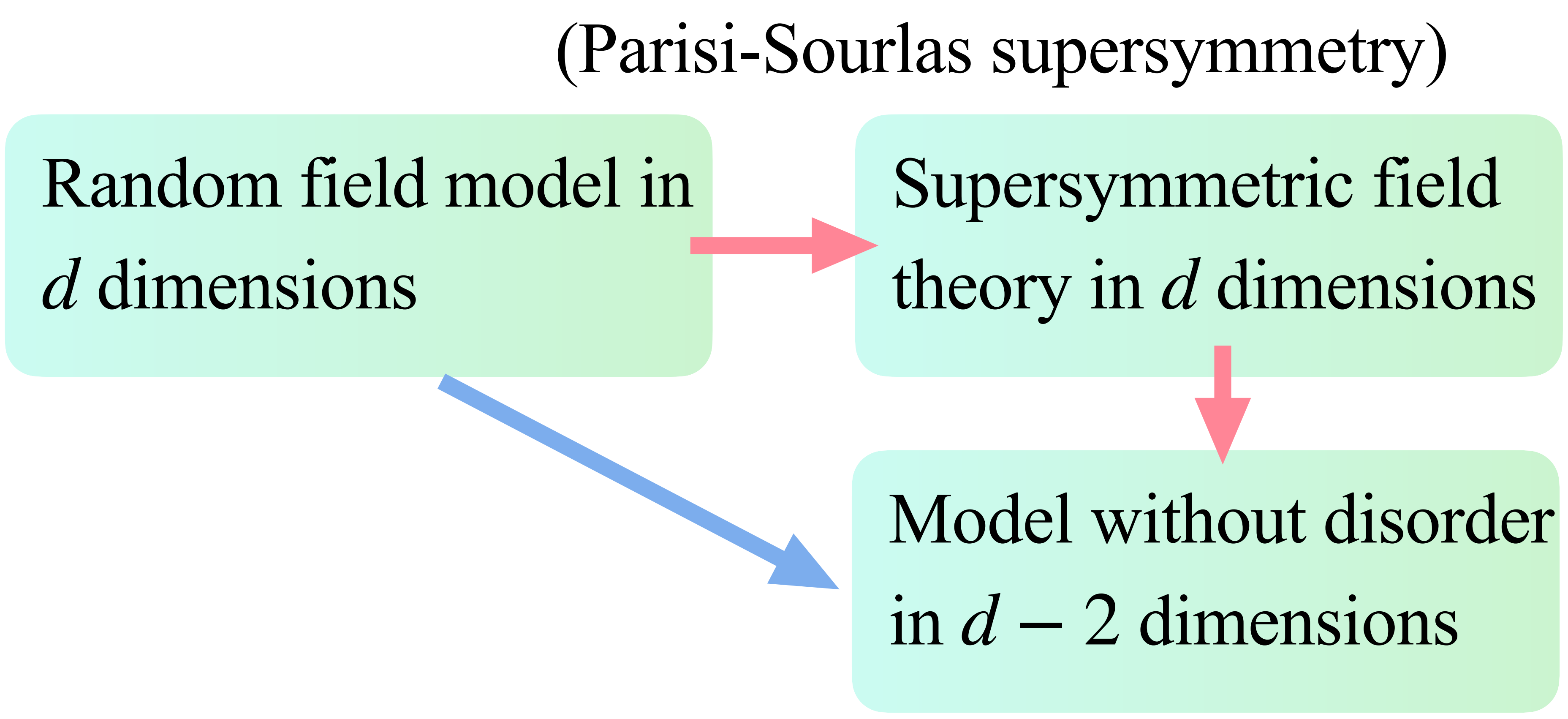
$M^{(s),d}_{\Delta} =$ 

Hidden structures

3. On the other hand, this combination of AdS_{d+1} diagrams is precisely just one scalar exchange Witten diagram in AdS_{d-3}!

$$\text{diff operator} \circ \underbrace{\left(M^{(s),d}_{\epsilon p} + \# M^{(s),d}_{\epsilon p+2} + \# M^{(s),d}_{\epsilon p+4}\right)}_{M^{(s),d-4}_{\epsilon p}}$$

Why should they equal, and what has this to do with Parisi-Sourlas?!



- Stochastic equations have a supersymmetry.
- Superspace: $(x, \theta, \bar{\theta})$, with $\mathfrak{osp}(d\,|\,2)$ symmetry.
- At fixed points, PS superconformal algebra: $\mathfrak{osp}(d+1,1\,|\,2)$.
- Supercharges don't obey spin statistics — unusual to high energy theorists.
- $\theta, \bar{\theta}$ effectively eat two directions of x, reducing the spacetime dimension.

Hidden structures

The PS supersymmetry was recently revisited from the conformal bootstrap perspective [Kaviraj, Rychkov, Trevisani], and led to interesting kinematic results. Among them is a beautiful formula for conformal blocks in different dimensions

$$g^{(d-2)}_{\Delta,\ell} = \underbrace{g^{(d)}_{\Delta,\ell} + C_{2,0}g^{(d)}_{\Delta+2,\ell} + C_{1,-1}g^{(d)}_{\Delta+1,\ell-1} + C_{0,-2}g^{(d)}_{\Delta,\ell-2} + C_{2,-2}g^{(d)}_{\Delta+2,\ell-2}}_{\text{superconformal block for } \mathfrak{osp}(d+1,1|2)}$$

The relation can be uplifted into AdS, and imply the following dimensional reduction relation for exchange Witten diagrams (imply the PS symmetry can be realized at tree-level in AdS) [XZ 2005.03031]

$$M^{(d-2)}_{\Delta,\ell} = M^{(d)}_{\Delta,\ell} + C_{2,0}M^{(d)}_{\Delta+2,\ell} + C_{1,-1}M^{(d)}_{\Delta+1,\ell-1} + C_{0,-2}M^{(d)}_{\Delta,\ell-2} + C_{2,-2}M^{(d)}_{\Delta+2,\ell-2}$$

($g_{\Delta,\ell} \to M_{\Delta,\ell}$ with appropriate contact terms)

Note it requires a nontrivial conspiracy of double-trace operators!

For $\ell = 0$, the relation becomes two-term

$$M^{(d-2)}_{\Delta,0} = M^{(d)}_{\Delta,0} + C_{2,0}M^{(d)}_{\Delta+2,0}$$

Hidden structures

By using the relation recursively for two times, we go from $d \to d-2 \to d-4$, and recover the precise combination of three scalar exchange diagrams!

$$\left.\begin{aligned} M^{(d-2)}_{\Delta,0} &= M^{(d)}_{\Delta,0} + C_{2,0} M^{(d)}_{\Delta+2,0} \\ M^{(d-4)}_{\Delta,0} &= M^{(d-2)}_{\Delta,0} + C_{2,0} M^{(d-2)}_{\Delta+2,0} \end{aligned}\right\} \quad \Rightarrow \quad \text{diff operator} \circ \underbrace{(M^{(s),d}_{\epsilon p} + \# M^{(s),d}_{\epsilon p+2} + \# M^{(s),d}_{\epsilon p+4})}_{M^{(s),d-4}_{\epsilon p}}$$

There is also a completely analogous reduction for the R-symmetry part, suggesting $S^q \to S^{q-4}$.

Therefore, we see the emergence of a new spacetime where both the AdS and the sphere have four less dimensions!

$$AdS_{d+1} \times S^q \xrightarrow[\;q \to q-4\;]{d \to d-4} AdS_{d-3} \times S^{q-4}$$

Hidden structures

This observation also gives a super compact way to rewrite our results.

$$S_p^{(s)} \propto C_{12p}C_{34p}\,\underline{K_p^{(s)}} \circ \left(\underline{Y_p^{(s),q-4}}\;\;\underline{M_p^{(s),d-4}}\right)$$

- $K_p^{(s)}$: A differential operator
- $Y_p^{(s),q-4}$: R-symmetry polynomial for $SO(q-3)$ of S^{q-4}
- $M_p^{(s),d-4}$: scalar exchange Witten diagram for AdS_{d-3}

This form of the results seems to suggest that formally there is a lower dimensional scalar "seed theory", which encodes all the data. The seed theory has the same spectrum as the superconformal primaries in the original theory, and the same cubic couplings. We can generate the full correlators from the seed correlators by acting with differential operators.

Twisted correlators

These holographic correlators contain a wealth of theory data. Let us focus on the protected information encoded in the "twisted correlator". Recall the superconformal Ward identity

$$(z\partial_z - \epsilon\alpha\partial_\alpha)\mathcal{G}(z,\bar{z};\alpha,\bar{\alpha})\Big|_{\alpha=1/z} = 0$$

It implies that

$d=6$: $\epsilon=2$, $\alpha=\bar{\alpha}=1/\bar{z}$, $\partial_{\bar{z}}\mathcal{G}(z,\bar{z};1/\bar{z},1/\bar{z})=0 \Rightarrow \mathcal{G}(z,\bar{z};1/\bar{z},1/\bar{z})=g(z)$ is meromorphic

$d=4$: $\epsilon=1$, $\bar{\alpha}=1/\bar{z}$, $\partial_{\bar{z}}\mathcal{G}(z,\bar{z};\alpha,1/\bar{z})=0 \Rightarrow \mathcal{G}(z,\bar{z};\alpha,1/\bar{z})=g(z,\alpha)$ is meromorphic

$d=3$: $\epsilon=1/2$, $z=\bar{z}=1/\bar{\alpha}$, $\partial_z\mathcal{G}(z,z;\alpha,1/z)=0 \Rightarrow \mathcal{G}(z,z;\alpha,1/z)=g(\bar{\alpha})$ is topological

The meromorphic and topological properties of the correlators in these special configurations can be systematically understood in terms of superconformal twisting (chiral algebra in 4d, 6d [Beem, Lemos, Liendo, Peelaers, Rastelli, van Rees '13; Beem, Rastelli, van Rees '14], TQM in 3d [Chester, Lee, Pufu, Yacoby '14; Beem, Rastelli, Peelaers '16]), which carves out a protected subsector in these theories.

Certain protected operators are restricted on a 2d plane or a line, with special R-symmetry polarizations determined by their location (twisting). One can then construct a nilpotent $\mathbb{Q}$, with respect to which the twisted operators are closed but not exact. The twisted translations are $\mathbb{Q}$-exact, making the correlators of twisted operators meromorphic/topological.

Twisted correlators

The meromorphic property can be explicitly checked in the holographic correlators by taking the residues and doing the twisting. The cancellation of the non-meromorphic piece requires contributions from all three channels, and provides a nontrivial consistency check for the results.

Moreover, the meromorphic correlators obtained from the supergravity calculation can also be compared with independent field theory calculations. This is because a meromorphic function is determined by its singularities, which are in turn controlled by singular OPE coefficients (i.e. 3-pt functions). We find perfect agreement in the two calculations, and therefore giving the chiral algebra conjectures a very nontrivial check at large central charges.

Checking the topological property for 3d is similar, but technically more difficult. From the supergravity result we can extract 3-pt coefficients in the protected subsector, which in principle can be compared with localization computations (subleading in $1/N$).

Conclusions & future directions

We presented a constructive derivation for all tree-level holographic four-point functions in theories with maximal superconformal symmetry. This method reproduces the bootstrap results for $AdS_5 \times S^5$, and gives new general results for $AdS_4 \times S^7$ and $AdS_7 \times S^4$.

- Our crucial observation is the MRV limit where amplitudes drastically simplify, as required by supersymmetry.
- The MRV amplitudes do not contain poles in the u-channel, have two zeroes, and also have improved u-channel Regge behavior. These simplifying features allow them to be easily computed.
- Studying MRV limit also reveals the underlying organizing principles of holographic correlators. Full correlators can be reconstructed from the MRV amplitudes by using only symmetries.
- A nice feature is that the Mellin amplitudes can be naturally written as the sum of exchange amplitudes, with no additional contact terms. It indicates certain on-shell constructiblilty. The absence of intrinsic contact terms is also observed at the level of the five-point function for stress tensor multiplets on $AdS_5 \times S^5$ [Goncalves, Pereira, XZ].
- In the results, we also saw a curious dimensional reduction structure, that allows us to compactly rewrite the amplitudes in terms only scalar amplitudes in lower dimensions.

Conclusions & future directions

- Extract CFT data and go to higher genus:
  - there is a wealth of protected/unprotected data which can be compared with localization and numerical bootstrap.
  - we have everything needed to construct one-loop amplitudes. The program is quite advanced for $AdS_5 \times S^5$ [Alday, Bissi; Aprile, Drummond, Heslop, Paul; Alday, Caron-Huot; Caron-Huot, Trinh; Alday, Bissi, Perlmutter; Alday, XZ; Bissi, Fardelli, Georgoudis ...], but still in its infancy for $AdS_7 \times S^4$ [Alday, Chester, Raj]. Nothing for $AdS_4 \times S^7$.
- Extension of MRV method to higher points and fewer supercharges
- Meaning of the Parisi-Sourlas type dimensional reduction?
  - Is it a coincidence for 4-pt? Or perhaps it is a property for n-pt? A more efficient way to compute n-pt?!
- More analogies with the flat space amplitudes:
  - Ideas and structures in flat space $\Rightarrow$ inspirations for AdS (MRV v.s. MHV in this talk).
  - absence of contact terms — can we find an holographic analogue of BCFW?
  - Other notions/structures? Color/kinematic duality [Armstrong, Lipstein, Mei; Albayrak, Kharel, Meltzer]? CHY formalism, ambitwistor strings [Eberhardt, Komatsu, Mizera; Roehrig, Siknner]?

Thank you for your attention!